

# Thermodynamics of rotating black holes with scalar hair in three dimensions

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Introducing a new form of scalar potential  $V(\phi)$ , we derive a proper form of the rotating black hole solution in three-dimensional Einstein gravity with nonminimally coupled scalar field and find that the first law of thermodynamics of this new rotating hairy black hole can be protected, where the scalar field parameter  $B$  is constrained to relate to the black hole size. We also disclose the Hawking-Page phase transition between this rotating hairy black holes and the pure thermal radiation. Moreover, we study phase transitions between this rotating hairy black hole and rotating BTZ black hole. Considering the matchings for the temperature and angular momentum, we find that the rotating BTZ black hole always has smaller free energy which is a thermodynamically more preferred phase. Additionally, we evaluate the thermodynamics of the rotating black hole with minimally coupled scalar hair in three dimensions, which exhibits that the thermodynamical behaviors of this rotating hairy black hole are very similar to those of the rotating black hole with nonminimally coupled scalar hair.

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## I. INTRODUCTION

Hairy black holes are interesting solutions in gravitational theories. They have been studied extensively over the past few years, mainly in connection with the no hair theorem. In general, the no-hair theorem rules out four-dimensional black holes coupled to scalar field in asymptotically flat spacetimes [1–3] because they are not physically acceptable since the scalar field can diverge on the horizon and black holes can become unstable [4]. In higher-dimensional cases ( $D > 4$ ), the hairy black hole solutions in asymptotically flat spacetimes simply do not exist [5, 6]. When the

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cosmological constant is taken into account, the no-hair theorem can usually be circumvented, and then some hairy black holes can be found and the divergence of the scalar field can be hidden behind the horizon. In asymptotically de Sitter (dS) spacetimes, hairy black hole minimally coupled to a scalar field were presented in Ref. [7], but it was found dynamically unstable. The exact black hole solutions with nonminimally coupled scalar field in dS spacetime have also been discussed in Refs. [8, 9], while they were also found unstable [10, 11].

When a negative cosmological constant is considered, four-dimensional (MTZ) black hole solutions with minimal [12, 13] and nonminimal [14] scalar fields can be obtained. Thermodynamically, the phase transitions between the MTZ black holes and black holes without scalar hair have been found of the second order both in the canonical ensemble [15] and the grand canonical ensemble [16]. The hairy black holes in anti-de Sitter(AdS) spaces were found stable by examining quasinormal modes in their backgrounds [17, 18]. Interestingly, the behavior of quasinormal modes was disclosed useful to reflect the thermodynamical second-order phase transition [17, 18]. More discussions on the hairy black hole solutions in AdS spacetimes can be found in Refs. [19–26]. Further explorations of the AdS hairy black holes in higher-curvature gravity theories have been reported in [27–30]. In three dimensional Einstein gravity, some static black holes with minimally [31] or nonminimally [20, 32] coupled scalar hairs were found. Moreover, the thermodynamical properties of these hairy black holes were studied in [15, 33, 34], where the first law of black hole thermodynamics was found always hold. It is interesting to note that the rotating black holes with nonminimally [35] and minimally [36] coupled scalar hairs in the three dimensional Einstein gravity have been recently also obtained. Examining the rotating black hole with non-minimally coupled scalar field in three dimensions [35], however, one can find that the thermodynamical properties of this rotating black hole were incorrectly presented in [37], where the first law of black hole thermodynamics cannot be protected when the parameter  $B$  of the scalar field is completely free. This is the first motivation of the present paper. On the other hand, the thermodynamical quantities  $M$  and  $J$  of this rotating hairy black hole also emerge in the expression for the scalar potential  $V(\phi)$  [35], which makes the action to be non-invariant. How to overcome these obstacles ? It is interesting to pursue.

In this article, we will first set a new form of the scalar potential  $V(\phi)$ , which refers to three parameters  $\mu$ ,  $\alpha$  and  $l$ , and obtain a proper form of the rotating black hole solution in the three-dimensional Einstein gravity with nonminimally coupled scalar field. After the discussion for the black hole horizon structures, the parameter  $B$  of the scalar field  $\phi(r)$  will be described to be related to the black hole horizon size through  $r_+ = \theta \times B$ , where  $\theta$  is a dimensionless constant.

Fortunately, it will show that this constraint on the parameter  $B$  is required to ensure the validity of the first law of thermodynamics for this rotating hairy black hole with nonminimally coupled scalar field. Furthermore, the phase transition between the three-dimensional rotating black hole with a nonminimally coupled scalar field and the rotating BTZ black hole will be discussed by using the temperature and angular momentum matchings. Besides, we will also extend these investigations to the rotating black hole with minimally coupled to scalar field in three dimensions [36], and discuss the phase transition between these rotating hairy and BTZ black holes.

This paper is organized as follows. In Sec. II, we will present a proper form of a rotating black hole solution in three-dimensional Einstein gravity with nonminimally coupled scalar field. In Sec. III, we will discuss the thermodynamical properties of this rotating hairy black hole and examine the first law of black hole thermodynamics. Moreover, by adopting temperature and angular momentum matchings, we will study the phase transition between the rotating hairy black hole and rotating BTZ black hole. In Sec. IV, we will extend the discussion to the rotating black hole with minimally coupled scalar hair. Finally, we will present conclusions and discussions in Sec. V.

## II. ROTATING BLACK HOLES NONMINIMALLY COUPLED TO SCALAR FIELD

We start with the action in three-dimensional Einstein gravity with a nonminimally coupled scalar field [35]

$$\mathcal{I} = \frac{1}{2} \int d^3x \sqrt{-g} (R - g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \xi R \phi^2 - 2V(\phi)), \quad (1)$$

where  $\xi$  equals to  $1/8$  signifying the coupling strength between gravity and the scalar field. Taking the scalar potential [35]

$$V(\phi) = -\frac{1}{l^2} + \frac{1}{512} \left( \frac{1}{l^2} + \frac{\beta}{B^2} \right) \phi^6 + \frac{1}{512} \left( \frac{a^2}{B^4} \right) \frac{(\phi^6 - 40\phi^4 + 640\phi^2 - 4608) \phi^{10}}{(\phi^2 - 8)^5}, \quad (2)$$

the rotating black hole solution is given by

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\psi + \omega(r)dt)^2, \quad (3)$$

$$f(r) = 3\beta + \frac{2B\beta}{r} + \frac{(3r + 2B)^2 a^2}{r^4} + \frac{r^2}{l^2}, \quad \omega(r) = -\frac{(3r + 2B)a}{r^3}, \quad (4)$$

and the scalar field takes the form

$$\phi(r) = \pm \sqrt{\frac{8B}{r+B}}. \quad (5)$$

Here, the parameters  $\beta$ ,  $B$ , and  $a$  are integration constants, and  $\Lambda = \frac{1}{l^2}$  appears in  $V(\phi)$  as a constant term, which plays the role of a (bare) cosmological constant.

The hairy black hole solution [Eq. (4)] can reduce to the rotating BTZ black hole solution [38] if one takes  $B \rightarrow 0$ , which leads to  $\beta = -\frac{M}{3}$  and  $a = \frac{J}{6}$ . The rotating black hole solution [Eq.(4)] can be rewritten into [35]

$$f(r) = -M \left( 1 + \frac{2B}{3r} \right) + \frac{r^2}{l^2} + \frac{(3r+2B)^2 J^2}{36r^4}, \quad \omega(r) = -\frac{(3r+2B) J}{6r^3}. \quad (6)$$

Nevertheless, it is necessary to point out that the action [Eq.(1)] is not invariant due to the existence of the thermodynamical quantities  $M$  and  $J$  of this rotating hairy black hole in the scalar potential  $V(\phi)$ .

The thermodynamical quantities of this rotating hairy black hole are [37]

$$M = \frac{J^2 l^2 (2B + 3r_+)^2 + 36r_+^6}{12l^2 r_+^3 (2B + 3r_+)}, \quad (7)$$

$$T = \frac{f'(r_+)}{4\pi} = \frac{(B + r_+) [36r_+^6 - J^2 l^2 (2B + 3r_+)^2]}{24\pi l^2 r_+^5 (2B + 3r_+)}, \quad (8)$$

$$S = \frac{A_H}{4G} (1 - \xi \phi^2(r_+)) = \frac{4\pi r_+^2}{B + r_+}, \quad (9)$$

$$\Omega_H = -\omega(r_+) = \frac{(3r_+ + 2B) J}{6r_+^3}, \quad (10)$$

where the parameter  $B$  of the scalar field is set to be arbitrary. One can also compute

$$\frac{\partial M}{\partial S} = \frac{(B + r_+)^3 [36r_+^6 - J^2 l^2 (2B + 3r_+)^2]}{8\pi l^2 r_+^5 (2B + 3r_+)^2 (2B + r_+)}, \quad (11)$$

which is different from Eq. (8), so that the first law of black hole thermodynamics meets the challenge in this rotating hairy black hole.

Now, we consider a new form of the scalar potential  $V(\phi)$

$$V(\phi) = -\frac{1}{l^2} + \frac{1}{512} \left( \frac{1}{l^2} + \mu \right) \phi^6 + \frac{\alpha^2 (\phi^6 - 40\phi^4 + 640\phi^2 - 4608) \phi^{10}}{512(\phi^2 - 8)^5}, \quad (12)$$

where  $\mu$ ,  $\alpha$ , and  $l$  are constant parameters. Performing the dimensional analysis, we have  $[l] = L$ ,  $[\mu] = L^{-2}$  and  $[\alpha] = L^{-1}$ . Moreover, the qualitative behavior of the scalar potential (Eq.(2)) has been adequately analyzed in [35]. Inserting Eq. (12) into the action, we can find the rotating hairy black hole with the metric coefficient

$$f(r) = \mu B^2 \left( 3 + \frac{2B}{r} \right) + \frac{(3r+2B)^2 \alpha^2 B^4}{r^4} + \frac{r^2}{l^2}, \quad \omega(r) = -\frac{\alpha B^2 (3r+2B)}{r^3}. \quad (13)$$

Now, when  $\alpha \rightarrow 0$ , the scalar potential  $V(\phi)$  and rotating black hole solution  $f(r)$  reduce to the static counterparts of the static black hole solution in three-dimensional Einstein gravity with a nonminimally coupled scalar field [20].

To see the horizon structures of this proper form of rotating hairy black hole, we define

$$X = \frac{3r_+ + 2B}{r_+^3}, \quad (14)$$

so that  $f(r_+) = 0$  gives

$$\alpha^2 B^4 X^2 + \mu B^2 X + \frac{1}{l^2} = 0, \quad (15)$$

which leads to

$$\begin{aligned} X_1 &= \frac{-\mu - \sqrt{\mu^2 - 4\alpha^2/l^2}}{2\alpha^2 B^2} = \frac{\tilde{X}_1}{B^2}, \\ X_2 &= \frac{-\mu + \sqrt{\mu^2 - 4\alpha^2/l^2}}{2\alpha^2 B^2} = \frac{\tilde{X}_2}{B^2}. \end{aligned} \quad (16)$$

Here  $X_\nu$ , ( $\nu = 1, 2$ ) are both real positive when  $-\mu \geq \frac{2\alpha}{l}$ , but they are both imaginary when  $-\mu < \frac{2\alpha}{l}$ . We focus on the case in which  $-\mu \geq \frac{2\alpha}{l}$ ; thus,  $\tilde{X}_\nu > 0$ .

Equating  $X$  to  $X_\nu$ , Eq. (14) becomes

$$H_\nu(r_+) = X_\nu r_+^3 - 3r_+ - 2B = \tilde{X}_\nu r_+^3 - 3B^2 r_+ - 2B^3 = 0. \quad (17)$$

The only positive solution of Eq. (17) exists when  $\frac{108B^6(\tilde{X}_\nu-1)}{\tilde{X}_\nu^3} > 0$ , namely  $\tilde{X}_\nu \geq 1$ , which is given by

$$r_+^\nu = \frac{1}{\tilde{X}_\nu} \left[ \left( \tilde{X}_\nu^2 - \sqrt{(\tilde{X}_\nu - 1)\tilde{X}_\nu^3} \right)^{1/3} + \left( \tilde{X}_\nu^2 + \sqrt{(\tilde{X}_\nu - 1)\tilde{X}_\nu^3} \right)^{1/3} \right] B = \hat{\theta}(\tilde{X}_\nu) B. \quad (18)$$

When  $\frac{108B^6(\tilde{X}_\nu-1)}{\tilde{X}_\nu^3} < 0$ , namely,  $0 < \tilde{X}_\nu < 1$ , there exist three roots for the solution of Eq. (17), but we only have interest in the positive one

$$r_+^\nu = \frac{B}{\sqrt{\tilde{X}_\nu}} \left( \cos \eta + \sqrt{3} \sin \eta \right) = \bar{\theta}(\tilde{X}_\nu) B, \quad \eta = \frac{\arccos(-\sqrt{\tilde{X}_\nu})}{3}. \quad (19)$$

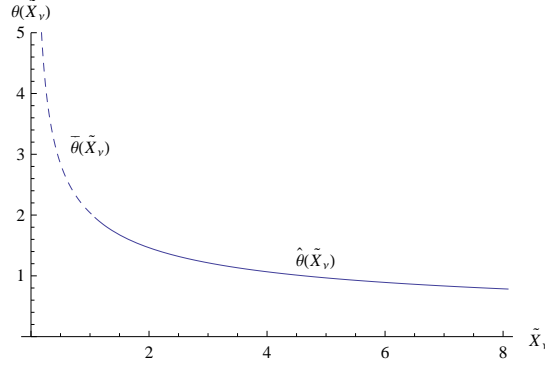
From the horizon structure, we find the constraint condition for the parameter of the scalar field  $B$ ,  $r_+^\nu = \theta(\tilde{X}_\nu) B$  when  $\tilde{X}_\nu > 0$ . Then  $\theta(\tilde{X}_\nu)$  is a dimensionless constant and the parameter  $B$  is no longer free. The coupling constant  $\theta(\tilde{X}_\nu)$  is plotted in Fig.1. For simplicity, in the following we write the constraint relation into

$$r_+ = \theta \times B. \quad (20)$$

The scalar field  $\phi(r)$  at the black hole horizon becomes

$$\phi(r_+) = \pm \sqrt{\frac{8B}{r_+ + B}} = \pm \sqrt{\frac{8}{1 + \theta}}, \quad (21)$$

which is independent of the horizon radius  $r_+$ .

FIG. 1:  $\theta(\tilde{X}_\nu)$  vs.  $\tilde{X}_\nu$ .

### III. THERMODYNAMICS OF ROTATING BLACK HOLE WITH NONMINIMAL SCALAR HAIR

In this rotating hairy black hole, the mass and angular momentum can be calculated by adopting the Brown-York method [39]. The quasilocal mass  $m(r)$  at  $r$  takes the form [40–42]

$$m(r) = \sqrt{f(r)}E(r) - j(r)\omega(r), \quad (22)$$

where  $E(r) = 2(\sqrt{f_0(r)} - \sqrt{f(r)})$  is the quasilocal energy at  $r$  and  $j(r) = \frac{d\omega(r)}{dr}r^3$  is the quasilocal angular momentum. Here,  $f_0(r)$  is a background metric function that determines the zero of the energy. By setting  $B = Q = 0$ , we can obtain the background metric  $f_0(r) = \frac{r^2}{l^2}$ . As a result, the mass and angular momentum of the black hole can be obtained as

$$M \equiv \lim_{r \rightarrow \infty} m(r) = -3\mu B^2, \quad J \equiv \lim_{r \rightarrow \infty} j(r) = 6\alpha B^2. \quad (23)$$

Thus the black hole metric coefficient [Eq. (13)] can be rewritten into

$$f(r) = -M \left( 1 + \frac{2B}{3r} \right) + \frac{r^2}{l^2} + \frac{(3r + 2B)^2 J^2}{36r^4}, \quad \omega(r) = -\frac{(3r + 2B) J}{6r^3}. \quad (24)$$

The condition  $-\mu \geq \frac{2\alpha}{l}$  is needed to protect the cosmic censorship, which puts the constraint  $\frac{M}{J} \geq \frac{1}{l}$  for this black hole.

#### A. First law of black hole thermodynamics

With Eq. (20), the mass of the rotating hairy black hole reads

$$\begin{aligned} M &= \frac{J^2 l^2 (2B + 3r_+)^2 + 36r_+^6}{12l^2 r_+^3 (2B + 3r_+)} \\ &= \frac{J^2 l^2 (2 + 3\theta)^2 + 36r_+^4 \theta^2}{12l^2 r_+^2 \theta (2 + 3\theta)}, \end{aligned} \quad (25)$$

and the entropy is

$$\begin{aligned} S &= \frac{A_H}{4G} (1 - \xi \phi^2(r_+)) \\ &= \frac{4\pi r_+^2}{B + r_+} = \frac{4\pi \theta r_+}{1 + \theta}. \end{aligned} \quad (26)$$

The temperature of this rotating hairy black hole can be derived as

$$\begin{aligned} T &= \frac{f'(r_+)}{4\pi} = \frac{(B + r_+) [36r_+^6 - J^2 l^2 (2B + 3r_+)^2]}{24\pi l^2 r_+^5 (2B + 3r_+)} \\ &= \frac{(1 + \theta) [36r_+^4 \theta^2 - J^2 l^2 (2 + 3\theta)^2]}{24\pi l^2 r_+^3 \theta^2 (2 + 3\theta)}. \end{aligned} \quad (27)$$

For  $T = 0$ , we can obtain the radius of extremal rotating hairy black hole  $r_{ext} = \left[ \frac{Jl(2+3\theta)}{6\theta} \right]^{1/2}$ .

In addition, there exist two commuting Killing vector fields for the metric [Eq. (3)]

$$\xi_{(t)} = \frac{\partial}{\partial t}, \quad \xi_{(\psi)} = \frac{\partial}{\partial \psi}. \quad (28)$$

The various scalar products of these Killing vectors can be expressed through the metric components

$$\begin{aligned} \xi_{(t)} \cdot \xi_{(t)} &= g_{tt} = -f(r), \\ \xi_{(t)} \cdot \xi_{(\psi)} &= g_{t\psi} = r^2 \omega(r), \\ \xi_{(\psi)} \cdot \xi_{(\psi)} &= g_{\psi\psi} = r^2. \end{aligned} \quad (29)$$

To examine physical processes near such a black hole, we introduce a family of locally nonrotating observers. The angular velocity for these observers that move on orbits with constant  $r$  and with a 4-velocity  $u^\mu$  satisfying  $u \cdot \xi_{(\psi)} = 0$  is given by [43, 44]

$$\Omega = -\frac{g_{t\psi}}{g_{\psi\psi}} = -\omega(r) = \frac{(3r + 2B) J}{6r^3}. \quad (30)$$

When approaching the black hole horizon, the angular velocity  $\Omega_H$  turns out to be

$$\begin{aligned} \Omega_H &= -\omega(r_+) = \frac{(3r_+ + 2B) J}{6r_+^3} \\ &= \frac{(2 + 3\theta) J}{6\theta r_+^2}. \end{aligned} \quad (31)$$

Combining these quantities,  $M$ ,  $T$ , and  $\Omega_H$ , we can verify that the first law of thermodynamics holds in this case,

$$dM = TdS + \Omega_H dJ \quad (32)$$

and the Smarr relation can be found,

$$M - \Omega_H J = \frac{1}{2} T S. \quad (33)$$

In general, the local thermodynamic stability is determined by the specific heat. The positive specific heat guarantees the stability of the black hole. When the specific heat becomes negative, it indicates that the black hole will be destroyed when it encounters a small perturbation. The specific heat can be calculated through  $C_J = T \left( \frac{\partial S}{\partial T} \right)_J$ , which reads

$$C_J = \frac{32\pi^2 l^2 r_+^4 \theta^3 (2 + 3\theta) T}{(1 + \theta)^2 \left[ (2 + 3\theta)^2 J^2 l^2 + 12 r_+^4 \theta^2 \right]}. \quad (34)$$

Apparently,  $C$  is always positive when  $T > 0$ , which implies that the rotating hairy black hole is locally stable when  $r_+ > r_{ext}$ .

### B. Phase transition between the rotating black hole with nonminimal scalar hair and rotating BTZ black hole

Now we examine the behavior of free energy  $F$ , which can be calculated as

$$F = M - T S = \frac{J^2 l^2 (2 + 3\theta)^2 - 12 r_+^4 \theta^2}{4 l^2 r_+^2 \theta (2 + 3\theta)}. \quad (35)$$

When the black hole horizon  $r_+ = r_c = \left( \frac{Jl(2+3\theta)}{2\sqrt{3\theta}} \right)^{1/2}$ , the free energy vanishes. At  $r_c$ , the black hole temperature  $T = T_c$ , where

$$T_c = \frac{J(1 + \theta)}{3^{1/4} l \pi} \sqrt{\frac{\theta}{2(2 + 3\theta) J l}}. \quad (36)$$

The critical temperature  $T_c$  depends on the values of  $\theta$ . The  $F - T$  relation with different values of  $\theta$  is plotted in Fig. 2, where we see that  $T_c$  increases with the growth of  $\theta$ . Moreover,  $F$  changes its sign at temperature  $T_c$ . In the region of  $0 < T < T_c$ , the free energy of this hairy black hole is always positive so the black hole phase evaporates completely to a pure thermal radiation phase. When  $T > T_c$ , the free energy of this hairy black hole will be less than that of pure thermal radiation, and then this hairy black hole will be thermodynamically favored. Hence there exist the Hawking-page phase transition between the black hole with nonminimally coupled scalar hair and the pure thermal radiation [45].

It is worth comparing the free energies for the hairy rotating black hole and the BTZ black hole. It is necessary to point out that, for a fixed value of  $M$ , from Eq. (24) we see that the scalar



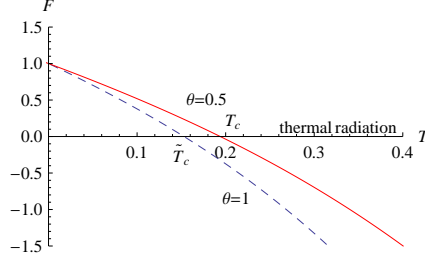


FIG. 2: The free energy  $F$  of the rotating hairy black hole relates to the temperature  $T$  for different  $\theta$  when we fix  $J = 1$  and  $l = 1$ .

field parameter  $B$  cannot be switched off naively to reduce the hairy black hole into the BTZ black hole. The rotating hairy black hole and the BTZ black hole cannot be smoothly deformed into each other; they belong to different disconnected phases.

In the limit  $\phi = 0$ , the scalar potential  $V(\phi)$  reduces to  $-\frac{1}{l^2}$ , and the action, Eq.(1), admits the rotating BTZ black hole [38]

$$ds^2 = - \left( -\hat{M} + \frac{\rho^2}{l^2} + \frac{\hat{J}^2}{4\rho^2} \right) dt^2 + \left( -\hat{M} + \frac{\rho^2}{l^2} + \frac{\hat{J}^2}{4\rho^2} \right)^{-1} d\rho^2 + \rho^2 \left( d\psi - \frac{\hat{J}}{2\rho^2} dt \right)^2. \quad (37)$$

The thermodynamic quantities, such as the temperature  $\hat{T}$ , mass  $\hat{M}$ , entropy  $\hat{S}$ , and free energy  $\hat{F}$  of the rotating BTZ black hole are given by

$$\begin{aligned} \hat{M} &= \frac{\rho_+^2}{l^2} + \frac{\hat{J}^2}{4\rho_+^2}, \quad \hat{T} = \frac{4\rho_+^4 - \hat{J}^2 l^2}{8\pi l^2 \rho_+^3}, \quad \hat{S} = 4\pi \rho_+, \\ \hat{F} &= \frac{3\hat{J}^2 l^2 - 4\rho_+^4}{4l^2 \rho_+^2}, \quad \Omega_H = \frac{\hat{J}}{2\rho_+^2}. \end{aligned} \quad (38)$$

To compare the free energies of the hairy black hole and the BTZ black hole, we need to match the temperature  $T = \hat{T}$  and the angular momentum  $J = \hat{J}$  of these two black holes,

$$\frac{(1+\theta) [36r_+^4 \theta^2 - J^2 l^2 (2+3\theta)^2]}{24\pi l^2 r_+^3 \theta^2 (2+3\theta)} = \frac{4\rho_+^4 - J^2 l^2}{8\pi l^2 \rho_+^3}. \quad (39)$$

In Fig. 3(a), we plot the free energies  $F$  of rotating hairy and BTZ black holes for different values of  $\theta$ . We find that the free energies  $F$  of rotating hairy black holes are always larger than that of the rotating BTZ black hole when  $T > 0$ . This means that thermodynamically the rotating BTZ black hole phase is more thermodynamically preferred. There exists a possible thermodynamical phase transition for the hairy black hole to become a rotating BTZ black hole, provided that there are some thermal fluctuations. On the other hand, we also evaluate the free energies of both black holes for different values of  $J$ ; see Fig.3(b). The free energy  $\hat{F}$  of a rotating BTZ black hole is always larger than that of the rotating hairy black hole. It implies that the relationship between

the free energies of the rotating BTZ and hairy black holes is not affected by the values of the angular momentum  $J$ .

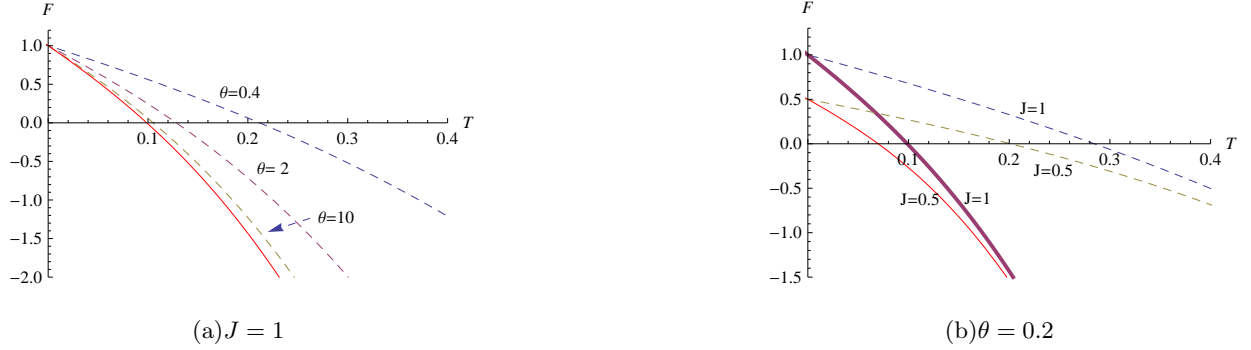


FIG. 3: The free energies  $F$  of (2+1)-dimensional rotating hairy black hole (dashed lines) and BTZ black hole (solid lines) vs the temperature  $T$  with  $l = 1$ .

#### IV. THERMODYNAMICS OF ROTATING BLACK HOLE WITH MINIMALLY COUPLED SCALAR HAIR

In this section, we generalize the above discussions to the three-dimensional rotating black hole with minimally coupled scalar hair. This black hole was obtained from the action [36]

$$I = \frac{1}{2} \int d^3x \sqrt{-g} (R - \nabla_\mu \phi \nabla^\mu \phi - 2V(\phi)), \quad (40)$$

where  $\phi(r)$  is the scalar field and the scalar potential takes the form

$$\begin{aligned} V(\phi) = & -\frac{1}{l^2} \cosh^6 \left( \frac{1}{2\sqrt{2}} \phi \right) + \frac{1}{l^2} (1 + \mu l^2) \sinh^6 \left( \frac{1}{2\sqrt{2}} \phi \right) \\ & - \frac{\alpha^2}{64} \sinh^{10} \left( \frac{1}{2\sqrt{2}} \phi \right) \cosh^6 \left( \frac{1}{2\sqrt{2}} \phi \right) \left[ \tanh^6 \left( \frac{1}{2\sqrt{2}} \phi \right) \right. \\ & \left. - 5 \tanh^4 \left( \frac{1}{2\sqrt{2}} \phi \right) + 10 \tanh^2 \left( \frac{1}{2\sqrt{2}} \phi \right) - 9 \right] \end{aligned} \quad (41)$$

with the constant parameters  $l$ ,  $\mu$ , and  $\alpha$ . Then, the rotating hairy black hole solution is obtained as [36]

$$\begin{aligned} ds^2 = & - \left( \frac{H(r)}{H(r) + B} \right)^2 f(H(r)) dt^2 + \left( \frac{H(r) + B}{H(r) + 2B} \right)^2 \frac{dr^2}{f(H(r))} \\ & + r^2 \left( d\psi - \omega(H(r)) dt \right)^2, \end{aligned} \quad (42)$$

$$f(H(r)) = 3\mu B^2 + \frac{2\mu B^3}{H(r)} + \frac{\alpha^2 B^4 (3H(r) + 2B)^2}{H(r)^4} + \frac{H(r)^2}{l^2}, \quad (43)$$

$$\omega(H(r)) = \frac{\alpha B^2 (3H(r) + 2B)}{H(r)^3}, \quad H(r) = \frac{1}{2} \left( r + \sqrt{r^2 + 4Br} \right) \quad (44)$$

and the scalar field is described by

$$\phi(r) = 2\sqrt{2}\operatorname{arctanh}\sqrt{\frac{B}{H(r)+B}}. \quad (45)$$

The event horizon is located at [36]

$$H(r_+) = h \times B, \quad (46)$$

where the parameter  $h$  only takes the real positive value of the three roots of  $f(H(r_+)) = 0$

$$\begin{aligned} h^{(1)} &= \frac{X_1}{\tilde{X}_0^{(i)}} + \frac{1}{X_1}, \\ h^{(2,3)} &= -\frac{1}{2}\left(\frac{X_1}{\tilde{X}_0^{(i)}} + \frac{1}{X_1}\right) \pm I\frac{\sqrt{3}}{2}\left(\frac{X_1}{\tilde{X}_0^{(i)}} - \frac{1}{X_1}\right) \end{aligned} \quad (47)$$

with

$$X_1 = \left[ \left( 1 + \sqrt{\frac{-1 + \tilde{X}_0^{(i)}}{\tilde{X}_0^{(i)}}} \right) \left( \tilde{X}_0^{(i)} \right)^2 \right]^{1/3}, \quad (i = 1, 2), \quad (48)$$

$$X_0^{(1)} = -\frac{\mu\ell + \sqrt{\mu^2\ell^2 - 4\alpha^2}}{2\alpha^2\ell B^2} = \frac{\tilde{X}_0^{(1)}}{B^2}, \quad (49)$$

$$X_0^{(2)} = -\frac{\mu\ell - \sqrt{\mu^2\ell^2 - 4\alpha^2}}{2\alpha^2\ell B^2} = \frac{\tilde{X}_0^{(2)}}{B^2}. \quad (50)$$

We require  $\mu \leq -\frac{2\alpha}{\ell}$  to satisfy  $X_0^i > 0$ , ( $i = 1, 2$ ).

The thermodynamical quantities of the rotating black hole with minimally coupled scalar field are given by [36]

$$\begin{aligned} M &= \frac{J^2 l^2 (3h+2)^2 + 36h^2 H(r_+)^4}{12l^2 h (3h+2) H(r_+)^2}, \quad T = \frac{(1+h) [36H(r_+)^4 h^2 - J^2 l^2 (2+3h)^2]}{24\pi l^2 h^2 (2+3h) H(r_+)^3}, \\ S &= \frac{4\pi h H(r_+)}{(h+1)}, \quad \Omega_H = \frac{(3h+2)J}{6hH(r_+)^3}, \quad F = \frac{J^2 l^2 (3h+2)^2 - 12h^2 H(r_+)^4}{4l^2 h (3h+2) H(r_+)^2}. \end{aligned} \quad (51)$$

These expressions are similar to their counterparts [Eqs. (25), (26), (27), (31), and (35)] for the rotating black hole nonminimally coupled with scalar field. It is easily found that the first law of black hole thermodynamics is well protected in this black hole background. Moreover, the free energy  $F$  of this rotating hairy black hole changes its sign at the critical temperature  $T_c = \frac{J(1+h)}{3^{1/4}l\pi} \sqrt{\frac{h}{2(2+3h)Jl}}$ . It is argued in Ref. [45] that this rotating hairy black hole is thermodynamically favored in the region of  $T > T_c$ , and when  $0 < T < T_c$ , this rotating hairy black hole phase evaporates completely into the pure thermal radiation. Therefore, there exists the so-called Hawking-Page phase transition between the black hole with the minimally coupled scalar field phase and the pure thermal radiation phase.

Following the above discussions, we can further examine the phase transitions between the rotating BTZ and this kind of rotating hairy black holes for the same temperature and angular momentum values:

$$\frac{(1+h) [36H(r_+)^4 h^2 - J^2 l^2 (2+3h)^2]}{24\pi l^2 h^2 (2+3h) H(r_+)^3} = \frac{4\rho_+^4 - J^2 l^2}{8\pi l^2 \rho_+^3}. \quad (52)$$

We find that the similar phenomena occur for these free energies; see Fig. 4. Here, the rotating BTZ black hole always possesses smaller free energy than the rotating black hole with a minimally coupled scalar field, which means that the rotating BTZ black hole is a more thermodynamically preferred phase. The thermodynamical behavior we observed here is very similar to the rotating black hole counterpart with nonminimally coupled scalar field.

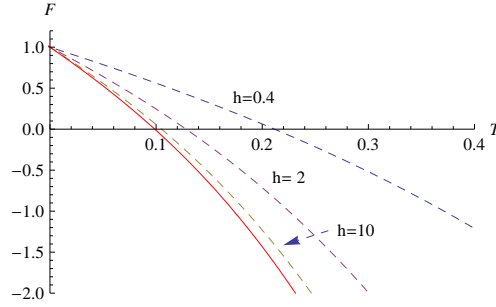


FIG. 4: The free energies  $F$  of (2+1)-dimensional rotating black hole with minimally scalar field (dashed lines) and BTZ black hole (solid line) vs the temperature  $T$  when  $l = 1$ .

## V. CONCLUSIONS AND DISCUSSIONS

In the backgrounds of three-dimensional rotating black holes with a nonminimally coupled scalar field, we found that when the scalar field parameter  $B$  is arbitrary, the first law of thermodynamics cannot be protected. By adopting a proper form of scalar potential  $V(\phi)$ , a proper form of rotating black hole solution in the three-dimensional Einstein gravity with a nonminimally coupled scalar field have been obtained, in which the scalar field parameter  $B$  is constrained to relate to the black hole size  $r_+$ , and the first law of thermodynamics of black hole is satisfied in this case.

We have further calculated the free energy in the three dimensional rotating hairy black hole backgrounds. Comparing the free energies of this rotating hairy black hole and the pure AdS space, we found that the Hawking-Page phase transition exists between this rotating hairy black hole and the pure AdS space. Moreover, we have computed the free energies for the rotating hairy black hole and the rotating BTZ black hole when they have the same temperature and angular

momentum. We disclosed that the rotating BTZ black hole has smaller free energy which is a thermodynamically more preferred phase. This property is general no matter whether the rotating black hole is minimally coupled or nonminimally coupled to scalar field.

In addition, some new rotating hairy black hole solutions have been recently found, such as the charged hairy black hole [46, 47], the Born-Infeld hairy black hole [48], the rotating charged hairy black hole with infinitesimal electric charge and rotation parameters [49], and black hole dressed by a (non)minimally coupled scalar field in new massive gravity [50], etc. It would be interesting to extend our discussion to explore the thermodynamical properties and phase transitions of these new black hole solutions and see whether the results obtained here are general.

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